Journal of
Linear and Topological Algebra
Vol. 08, *No.* 01, 2019, 53- 61

On a generalization of central Armendariz rings

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Received 1 September 2018; Revised 18 November 2018; Accepted 30 November 2018.
Communicated by Hamidreza Rahimi

Abstract. In this paper, some properties of α-skew Armendariz and central Armendariz rings have been studied by variety of others. We generalize the notions to central α-skew Armendariz rings and investigate their properties. Also, we show that if $\alpha(e) = e$ for each idempotent $e^2 = e \in R$ and R is α-skew Armendariz, then R is abelian. Moreover, if R is central α-skew Armendariz, then R is right p.p-ring if and only if $R[x;\alpha]$ is right p.p-ring. Then it is proved that if $\alpha^t = I_R$ for some positive integer t, R is central α-skew Armendariz if and only if the polynomial ring $R[x]$ is central α-skew Armendariz if and only if the Laurent polynomial ring $R[x, x^{-1}]$ is central α-skew Armendariz.

Keywords: α-skew Armendariz rings, central Armendariz rings, central α-skew Armendariz rings.

2010 AMS Subject Classification: 16U20, 16S36.

1. Introduction and preliminaries

Throughout this article, R is an associative ring with identity. The center of a ring R and the set of all the units in R are denoted by $C(R)$ and $U(R)$, respectively. In 1997, Rege and Chhawchharia [10] introduced the notion of an Armendariz ring. They called a ring R an Armendariz ring if whenever nonzero polynomials $f(x) = a_0 + a_1x + \cdots + a_nx^n$ and $g(x) = b_0 + b_1x + \cdots + b_mx^m \in R[x]$ satisfy $f(x)g(x) = 0$, then $a_ib_j = 0$ for all i and j. The name "Armendariz ring" is chosen because Armendariz [3, Lemma 1] has been shown that

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reduced ring (that is, a ring without nonzero nilpotent) satisfies this condition. A number of properties of the Armendariz rings have been studied in [2–4, 8–10]. So far, Armendariz rings are generalized in several forms. Let α be an endomorphism of a ring R. In 2003, Hong et al. [7] introduced a possible generalization of the Armendariz rings. A ring R is called α-Armendariz if for any $f(x) = \sum_{i=0}^{n} a_i x^i$ and $g(x) = \sum_{j=0}^{m} b_j x^j$ in $R[x;\alpha]$, $f(x)g(x) = 0$ implies that $a_i b_j = 0$ for all i, j. According to [6], a ring R is called α-skew Armendariz if $f(x)g(x) = 0$ for $f(x) = \sum_{i=0}^{n} a_i x^i$ and $g(x) = \sum_{j=0}^{m} b_j x^j \in R[x;\alpha]$, then $a_i \alpha^i(b_j) = 0$ for all i, j. They showed that if a ring R is α-rigid (that is, if $a\alpha(a) = 0$ then $a = 0$ for $a \in R$), then $R[x]/\langle x^2 \rangle$ is $\bar{\alpha}$-skew Armendariz. They also showed that if $\alpha^t = I_R$ for some positive integer t, then R is α-skew Armendariz if and only if $R[x]$ is α-skew Armendariz. Agayev et al. [1] called a ring R central Armendariz if whenever polynomials $f(x) = a_0 + a_1 x + \cdots + a_n x^n$ and $g(x) = b_0 + b_1 x + \cdots + b_m x^m \in R[x]$ satisfy $f(x)g(x) = 0$, then $a_i b_j \in C(R)$ for all i, j. They showed that the class of central Armendariz rings lies precisely between classes of Armendariz rings and abelian rings (that is, its idempotents belong to $C(R)$). For a ring R, they proved that R is central Armendariz if and only if $R[x]$ is central Armendariz if and only if $R[x, x^{-1}]$ is central Armendariz, where $R[x]$ is the polynomial ring and $R[x, x^{-1}]$ is the Laurent polynomial ring over a ring R. Furthermore, they showed that if R is reduced, then $R[x]/\langle x^n \rangle$ is central Armendariz and the converse holds if R is semiprime, where $\langle x^n \rangle$ is the ideal generated by x^n and $n \geqslant 2$. Motivated by the above results, for an endomorphism α of a ring R, we investigate a generalization of the α-skew Armendariz rings and the central Armendariz rings.

2. Central α-skew Armendariz rings

In this section, the central α-skew Armendariz rings are introduced as a generalization of the α-skew Armendariz ring.

Definition 2.1 Let α be an endomorphism of a ring R. The ring R is called a central α-skew Armendariz ring if for any nonzero polynomial $f(x) = \sum_{i=0}^{n} a_i x^i$ and $g(x) = \sum_{j=0}^{m} b_j x^j \in R[x;\alpha]$, $f(x)g(x) = 0$ implies that $a_i \alpha^i(b_j) \in C(R)$ for each i, j.

Note that all commutative rings, α-skew Armendariz rings and the subrings of central α-skew Armendariz rings are central α-skew Armendariz. Also, since each reduced ring R is I_R-skew Armendariz, where I_R is an identity map, then each reduced ring is central I_R-skew Armendariz ring.

The following examples show that the central α-skew Armendariz rings are not necessary α-skew Armendariz.

***Example* 2.2** (1) Let $R = R_1 \oplus R_2$, where R_i is a commutative ring for $i = 1, 2$. Let $\alpha : R \longrightarrow R$ be an automorphism defined by $\alpha((a, b)) = (b, a)$, then for $f(x) = (1, 0) - (1, 0)x$ and $g(x) = (0, 1) + (1, 0)x$ in $R[x;\alpha]$, $f(x)g(x) = 0$, but $(1, 0)\alpha((0, 1)) = (1, 0)^2 \neq 0$. Therefore, R is not α-skew Armendariz. But R is central α-skew Armendariz, since R is commutative.
(2) Let $\mathbb{Z}_4$ be the ring of integers modulo 4. Consider a ring $R = \left\{ \left(\begin{smallmatrix} \bar{a} & \bar{b} \\ 0 & \bar{a} \end{smallmatrix} \right) \middle| \bar{a}, \bar{b} \in \mathbb{Z}_4 \right\}$ and $\alpha : R \longrightarrow R$ be an endomorphism defined by $\alpha\left(\left(\begin{smallmatrix} \bar{a} & \bar{b} \\ 0 & \bar{a} \end{smallmatrix} \right) \right) = \left(\begin{smallmatrix} \bar{a} & -\bar{b} \\ 0 & \bar{a} \end{smallmatrix} \right)$. The ring R is not α-skew Armendariz. In fact $\left(\left(\begin{smallmatrix} \bar{2} & \bar{0} \\ 0 & \bar{2} \end{smallmatrix} \right) + \left(\begin{smallmatrix} \bar{2} & \bar{1} \\ 0 & \bar{2} \end{smallmatrix} \right) x \right)^2 = 0 \in R[x;\alpha]$ but $\left(\begin{smallmatrix} \bar{2} & \bar{1} \\ 0 & \bar{2} \end{smallmatrix} \right) \alpha\left(\left(\begin{smallmatrix} \bar{2} & \bar{0} \\ 0 & \bar{2} \end{smallmatrix} \right) \right) = \left(\begin{smallmatrix} \bar{0} & \bar{2} \\ 0 & \bar{0} \end{smallmatrix} \right) \neq 0$. But it can be easily checked that R is commutative and so, it is central α-skew

Armendariz ring.

Let R_i be a ring and α_i an endomorphism of R_i for each $i \in I$. Then for endomorphism $\bar{\alpha} : \prod_{i\in I} R_i \longrightarrow \prod_{i\in I} R_i$ defined by $\bar{\alpha}(a_i)_{i\in I} = (\alpha_i(a_i))_{i\in I}$, $\prod_{i\in I} R_i$ is central α-skew Armendariz if and only if each R_i is central α-skew Armendariz.

Proposition 2.3 Let α be an endomorphism of a ring R, S be a ring and $\varphi : R \longrightarrow S$ be an isomorphism. Then R is central α-skew Armendariz if and only if S is central $\varphi\alpha\varphi^{-1}$-skew Armendariz.

Proof. Let $\alpha' = \varphi\alpha\varphi^{-1}$. Clearly, α' is an endomorphism of S. Suppose that $a' = \varphi(a)$ for $a \in R$. Note that $\varphi(a\alpha^i(b)) = a'\varphi(\alpha^i(b))$ for all $a, b \in R$. Also, $f(x) = \sum_{i=0}^{n} a_i x^i$ and $g(x) = \sum_{j=0}^{m} b_j x^j$ are nonzero in $R[x;\alpha]$ if and only if $f'(x) = \sum_{i=0}^{n} a'_i x^i$ and $g'(x) = \sum_{j=0}^{m} b'_j x^j$ are nonzero in $S[x;\alpha']$. On the other hand, $f(x)g(x) = 0$ in $R[x;\alpha]$ if and only if $f'(x)g'(x) = 0$ in $S[x;\alpha']$. Also, since φ is an isomorphism, $a'_i(\varphi\alpha\varphi)^i b'_j = a'_i\varphi\alpha^i\phi^{-1}(b'_j) = \varphi(a_i)\varphi\alpha^i(b_j) = \varphi(a_i\alpha^i(b_j)) \in C(S)$ if and only if $a_i\alpha^i(b_j) \in C(R)$. Thus, R is central α-skew Armendariz if and only if S is central $\varphi\alpha\varphi^{-1}$-skew Armendariz. ■

The following example shows that there exists a central α-skew Armendariz ring such that $\alpha(e) \neq e$ for some $e^2 = e \in R$.

***Example* 2.4** Let $R = \left\{ \left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right) \middle| a, b \in \mathbb{Z}_2 \oplus \mathbb{Z}_2 \right\}$, where $\mathbb{Z}_2$ is the ring of integers modulo 2. Let $\alpha : R \longrightarrow R$ be defined by $\alpha\Big(\left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right)\Big) = \left(\begin{smallmatrix} b & 0 \\ 0 & a \end{smallmatrix}\right)$. Then R is a commutative ring and so it is central α-skew Armendariz. But $\alpha(e) \neq e$ for $e = \left(\begin{smallmatrix} (1,0) & (0,0) \\ (0,0) & (0,1) \end{smallmatrix}\right)$.

Recall that a ring R is said to be abelian if every idempotent of R is central.

Proposition 2.5 Let R be a ring and α be an endomorphism with $\alpha(e) = e$ for any $e^2 = e \in R$. Then R is central α-skew Armendariz ring if and only if R is abelian and eR and $(1-e)R$ are central α-skew Armendariz for some $e^2 = e \in R$.

Proof. If R is central α-skew Armendariz, they eR and $(1-e)R$ are central α-skew Armendariz since they are the invariant subrings of R. Now, let e be an idempotent in R. Consider $f(x) = e - er(1-e)x$ and $g(x) = (1-e) + er(1-e)x$. Since $\alpha(e) = e$, we have $f(x)g(x) = 0$. By hypothesis, $er(1-e)$ is central and so, $er(1-e) = 0$. Hence, $er = ere$ for each $r \in R$. Similarly, consider $p(x) = (1-e) - (1-e)rex$ and $q(x) = e + (1-e)rex$ in $R[x;\alpha]$ for all $r \in R$. Then $p(x)q(x) = 0$. As before $(1-e)re = 0$ and $ere = re$ for all $r \in R$, it follows that e is central element of R; that is, R is abelian. Conversely, let $f(x) = \sum_{i=0}^{n} a_i x^i$ and $g(x) = \sum_{j=0}^{m} a_j x^j$ be nonzero polynomials in $R[x;\alpha]$ such that $f(x)g(x) = 0$. Let $f_1(x) = ef(x)$, $g_1(x) = eg(x)$, $f_2(x) = (1-e)f(x)$, $g_2(x) = (1-e)g(x)$. Then $f_1(x)g_1(x) = 0$ in $(eR)[x;\alpha]$ and $f_2(x)g_2(x) = 0$ in $((1-e)R)[x;\alpha]$. Since eR and $(1-e)R$ are central α-skew Armendariz, $ea_ie\alpha^i(b_j)$ is central in eR and $(1-e)a_i(1-e)\alpha^i(b_j)$ is central in $(1-e)R$. Since e and $1-e$ are central in R, $R = eR \oplus (1-e)R$ and so, $a_i\alpha^i(b_j) = ea_i\alpha^i(b_j) + (1-e)a_i\alpha^i(b_j)$ is central in R for all $0 \leqslant i \leqslant n$, $0 \leqslant j \leqslant m$. Therefore, R is central α-skew Armendariz. ■

Corollary 2.6 Let α be an endomorphism of a ring R with $\alpha(e) = e$ for any $e^2 = e \in R$. If R is α-skew Armendariz ring then R is abelian.

***Remark* 1** *[1, Example 2.2] shows that abelian rings need not to be central Armendariz in general. Clearly, for any ring R and endomorphism $\alpha = I_R$, the abelian rings in general are not central α-skew Armendariz.*

Lemma 2.7 Let R be a central α-skew Armendariz ring and e be an idempotent element in $R[x;\alpha]$. If $e = e_0 + e_1x + \cdots + e_nx^n$, then $e_i \in C(R)$ for $i = 1, 2, \cdots, n$. Moreover, if $\alpha(e) = e$, then $e = e_0$.

Proof. Since $e(1-e) = 0 = (1-e)e$, we have $(e_0 + e_1x + \cdots + e_nx^n)(1 - e_0 + e_1x + \cdots + e_nx^n) = 0$ and $(1 - e_0 + e_1x + \cdots + e_nx^n)(e_0 + e_1x + \cdots + e_nx^n) = 0$. Since R is a central α-skew Armendariz ring, $e_0e_i \in C(R)$ and $(1-e_0)e_i \in C(R)$ for $1 \leqslant i \leqslant n$. Thus, $e_i \in C(R)$ for $1 \leqslant i \leqslant n$. Now, let R is a central α-skew Armendariz ring and $\alpha(e) = e$. It follows from Proposition 2.5 that R is abelian. The rest follows from Theorem 2.9 in [5]. ■

Proposition 2.8 Let R be a central α-skew Armendariz ring. Then $R[x;\alpha]$ is abelian if and only if $\alpha(e) = e$ for any $e^2 = e \in R$.

Proof. Suppose that $R[x;\alpha]$ is abelian and $e^2 = e \in R[x;\alpha]$. Then e is central and so, $ex = xe = \alpha(e)x$. Thus, $\alpha(e) = e$. Conversely, let $\alpha(e) = e$ for any $e^2 = e \in R$. Since R is central α-skew Armendariz by Proposition 2.5, R is abelian. Now, let $e^2 = e \in R[x;\alpha]$. By Lemma 2.7, e is an idempotent in R. For any $p = a_0x^k + a_1x^{k+1} + \cdots + a_mx^{k+m} \in R[x;\alpha]$, where k and m are nonnegative integers, we have $pe = (a_0x^k + a_1x^{k+1} + \cdots + a_mx^{k+m})e = a_0\alpha^k(e)x^k + a_1\alpha^{k+}(e)x^{k+1} + \cdots + a_m\alpha^{k+m}(e)x^{k+m} = e(a_0x^k + a_1x^{k+1} + \cdots + a_mx^{k+m}) = ep$. since R is abelian and $\alpha(e) = e$, $R[x;\alpha]$ is abelian. ■

For a nonempty subset X of a ring R, we write $r_R(X) = \{r \in R | xr = 0 \ for\ any\ x \in X\}$, which is called right annihilator of X in R. Kaplansky [1] introduced Baer rings as rings in which the right annihilator of every nonempty subset is generated by an idempotent. As a generalization of Baer rings, a ring R is called a right (resp., left) *p.p*-ring if the right (resp., left) annihilator of an element of R is generated (as a right (resp., left) ideal) by an idempotent of R.

Theorem 2.9 For an endomorphism α of a ring R, if the ring R is α-skew Armendariz, then R is central α-skew Armendariz. The converse hold if R is right p.p-ring and $\alpha(e) = e$ for any $e^2 = e \in R$.

Proof. Clearly, α-skew Armendariz rings are central α-skew Armendariz. For converse, suppose R is a central α-skew Armendariz and right p.p-ring. Then by Proposition 2.5, R is abelian. Let $f(x) = \sum_{i=0}^{n} a_ix^i$ and $g(x) = \sum_{j=0}^{m} b_jx^j$, $f(x)g(x) = 0$. We have

$$a_0b_0 = 0 \tag{1}$$

$$a_0b_1 + a_1\alpha(b_0) = 0 \tag{2}$$

$$a_0b_2 + a_1\alpha(b_1) + a_2\alpha^2(b_0) = 0 \tag{3}$$

By hypothesis there exists idempotent $e_i \in R$ such that $r(a_i) = e_iR$, for all i. Therefore $b_0 = e_0b_0$ and $a_0e_0 = 0$. Multiplying (2) by e_0 from the right, then $0 = a_0b_1e_0 + a_1\alpha(b_0)e_0 = a_0e_0b_1 + a_1\alpha(b_0)e_0 = a_1\alpha(b_0)e_0$. Hence, $a_1\alpha(b_0) = a_1\alpha(b_0e_0) = 0$. By (2), $a_0b_1 = 0$ and so, $b_1 = e_0b_1$. Again multiplying (3) from the right by e_0, then

$$\begin{aligned} 0 &= a_0b_2e_0 + a_1\alpha(b_1)e_0 + a_2\alpha^2(b_0)e_0 \\ &= a_0b_2e_0 + a_1\alpha(b_1)\alpha(e_0) + a_2\alpha^2(b_0)\alpha^2(e_0) \\ &= a_0e_0b_2 + a_1\alpha(e_0b_1) + a_2\alpha^2(e_0b_0) \\ &= a_1\alpha(b_1) + a_2\alpha^2(b_0) \end{aligned}$$

Multiplying this equation from the right by e_1. Hence,

$$\begin{aligned}0 &= a_1\alpha(b_1)e_1 + a_2\alpha^2(b_0)e_1\\ &= a_1e_1\alpha(b_1) + a_2\alpha^2(b_0)e_1\\ &= a_2\alpha(\alpha(b_0))\alpha(e_1)\\ &= a_2\alpha(\alpha(b_0)e_1)\\ &= a_2\alpha(\alpha(b_0))\\ &= a_2\alpha^2(b_0).\end{aligned}$$

Continuing this process, we have $a_i\alpha^i(b_j) = 0$ for all $0 \leqslant i \leqslant n$ and $0 \leqslant j \leqslant m$. Thus, R is α-skew Armendariz. This completes the proof. ■

In [1, Example 2.3], it is shown that the hypothesis that R be right p.p-ring is essential in Theorem 2.9 for the endomorphism $\alpha = I_R$.

Corollary 2.10 Assume that α is an automorphism of a ring R with $\alpha(e) = e$ for any $e^2 = e \in R$. If R is a central α-skew Armendariz ring, then R is right p.p-ring if and only if $R[x;\alpha]$ is right p.p-ring.

Proof. Let R be right p.p-ring. By Theorem 2.9, R is α-skew Armendariz. So the proof is done by [6, Theorem 22]. Conversely, assume that $R[x;\alpha]$ is a right p.p-ring. Let $a \in R$. By Lemma 2.7, there exists an idempotent $e \in R$ such that $r_{R[x;\alpha]}(a) = eR[x;\alpha]$. Hence, $r_R(a) = eR$. Therefore R is a p.p-ring. ■

Recall that if α is an endomorphism of a ring R, then the map $\bar{\alpha} : R[x] \longrightarrow R[x]$ defined by $\bar{\alpha}(\sum_{i=0}^m a_ix^i) = \sum_{i=0}^m \alpha(a_i)x^i$ is an endomorphism of the polynomial ring $R[x]$ and clearly this map extends α. We shall also denote the extended map $R[x] \longrightarrow R[x]$ by α and the image of $f \in R[x]$ by $\alpha(f)$. Note that by [6, Theorem 6], a ring R is α-skew Armendariz if and only if $R[x]$ is α-skew Armendariz for an endomorphism α with $\alpha^t = I_R$ for some positive integer t. Similarly, we have the following result.

Theorem 2.11 Let α be an endomorphism of a ring R and $\alpha^t = I_R$ for some positive integer t. Then R is central α-skew Armendariz if and only if $R[x]$ is central α-skew Armendariz.

Proof. Assume that $R[x]$ is central α-skew Armendariz. Then R is central α-skew Armendariz as a subring of $R[x]$. Conversely, assume that R is central α-skew Armendariz. Suppose that $p(y) = f_0 + f_1y + \ldots + f_my^m$; $q(y) = g_0 + g_1y + \ldots + g_ny^n$ in $R[x][y;\alpha] - \{0\}$ and $p(y)q(y) = 0$. Also, let $f_i = a_{i_0} + a_{i_1}x + \ldots + a_{i\omega_i}x^{\omega_i}$ and $g_j = b_{j_0} + b_{j_1}x + \ldots + b_{j\nu_j}x^{\nu_j}$ for each $0 \leqslant i \leqslant m$ and $0 \leqslant j \leqslant n$, where $a_{i_0}, \ldots, a_{i\omega_i}, b_{j_0}, \ldots, a_{j\nu_j} \in R$. We claim that $f_i\alpha^i(g_j) \in C(R[x])$ for all $0 \leqslant i \leqslant m$ and $0 \leqslant j \leqslant n$. Take a positive integer k such that $k \geqslant deg(f_0(x)) + deg(f_1(x)) + \cdots + deg(f_m(x)) + deg(g_0(x)) + deg(g_1(x)) \cdots + deg(g_n(x))$, where the degree is as polynomials in $R[x]$ and the degree of zero polynomial is taken to be 0. Since $p(y)q(y) = 0$ in $R[x][y;\alpha] - \{0\}$, we have

$$\begin{cases} f_0(x)g_0(x) = 0\\ f_0(x)g_1(x) + f_1(x)\alpha(g_0(x)) = 0\\ \vdots\\ f_m(x)\alpha^m(g_n(x)) = 0\end{cases} \tag{4}$$

in $R[x]$. Now, put

$$\begin{aligned} f(x) &= f_0(x^t) + f_1(x^t)x^{tk+1} + f_2(x^t)x^{2tk+2} + \cdots + f_m(x^t)x^{mtk+m}, \\ g(x) &= g_0(x^t) + g_1(x^t)x^{tk+1} + g_2(x^t)x^{2tk+2} + \cdots + g_n(x^t)x^{ntk+n}. \end{aligned} \tag{5}$$

Note that $\alpha^t = I_R$. Then

$$f(x)g(x) = f_0(x^t)g_0(x^t) + (f_0(x^t)g_1(x^t) + f_1(x^t)\alpha(g_0(x^t)))x^{tk+1} + \cdots + f_m(x^t)\alpha^m(g_n(x^t))x^{m+n(tk+1)}$$

in $R[x;\alpha]$. Using (4) and $\alpha^t = I_R$, we have $f(x)g(x) = 0$ in $R[x;\alpha]$. On the other hand, from (5), we have

$$f(x)g(x) = (a_{00}+a_{01}x^t+\cdots+a_{0\omega_0}x^{\omega_0 t}+a_{10}x^{tk+1}+a_{11}x^{tk+t+1}+\cdots+a_{1\omega_1}x^{tk+\omega_1 t+1}+\cdots+ a_{m0}x^{mtk+m}+a_{m1}x^{mtk+t+m}+\cdots+a_{m\omega_m}x^{mtk+\omega_m t+m})(b_{00}+b_{01}x^t+\cdots+b_{0\nu_0}x^{\nu_0 t}+b_{10}x^{tk+1}+ b_{11}x^{tk+t+1}+\cdots+b_{1\nu_1}x^{tk+\nu_1 t+1}+\cdots+b_{n0}x^{ntk+n}+b_{n1}x^{ntk+t+n}+\cdots+b_{n\omega_n}x^{ntk+\nu_n t+n}) = 0.$$

Since R is central α-skew Armendariz and $\alpha^t = I_R$, then $a_{iu}\alpha^i(b_{jv}) = a_{iu}\alpha^{itk+ut+i}(b_{jv}) \in C(R)$ for each $0 \leqslant i \leqslant m$, $0 \leqslant j \leqslant n$, and $u \in \{0, 1, \ldots, \omega_i\}$, $v \in \{0, 1, \ldots, \nu_j\}$. Since $C(R)$ is closed under addition, we have $f_i(x^t)\alpha^i(g_j(x^t)) \in C(R[x])$ for every $0 \leqslant i \leqslant m$ and $0 \leqslant j \leqslant n$. Now, it is easy to see that $f_i(x)\alpha^i(g_j(x)) \in C(R[x])$. Hence, $R[x]$ is central α-skew Armendariz. ■

Let R be a ring. For any integer $n \geqslant 2$, consider the ring $M_n(R)$ of $n \times n$ matrices and the ring $T_n(R)$ of $n \times n$ triangular matrices over a ring R. The $n \times n$ identity matrix is denoted by I_n. For $n \geqslant 2$, let $\{e_{i,j|1\leqslant i,j\leqslant n}\}$ be the set of the matrix units. Let $\alpha : R \longrightarrow R$ be a ring endomorphism with $\alpha(1) = 1$. For any $A = (a_{i,j}) \in M_n(R)$, we denote $\bar{\alpha} : M_n(R) \longrightarrow M_n(R)$ by $\bar{\alpha}((a_{i,j})_{n\times n}) = (\alpha(a_{i,j}))_{n\times n}$. So $\bar{\alpha}$ is a ring endomorphism of the ring $M_n(R)$. The rings $M_n(R)$ and $T_n(R)$ contain non-central idempotents. Therefore, they are not abelian. By Proposition 2.5, these rings are not central I_R-skew Armendariz.

Now, we introduce a notation for some subring of $T_n(R)$ that will be central $\bar{\alpha}$-skew Armendariz.

Given a ring R and (R, R)-bimodule M, the trivial extension of R by M is the ring $T(R, M) = R \oplus M$ with the usual addition and the following multiplication:

$$(r_1, m_1)(r_2, m_2) = (r_1 r_2, r_1 m_2 + m_1 r_2).$$

This is isomorphic to the ring of all matrices $\left(\begin{smallmatrix} r & m \\ 0 & r \end{smallmatrix}\right)$, where $r \in R$ and $m \in M$ and usual matrix operations are used. For an endomorphism α of a ring R and the trivial extension $T(R, R)$ of R, $\bar{\alpha} : T(R, R) \longrightarrow T(R, R)$ defined by $\bar{\alpha}((a, b)) = (\alpha(a), \alpha(b))$ is an endomorphism of $T(R, R)$. Since $T(R, 0)$ and R are isomorphic, we can describe the restriction of $\bar{\alpha}$ by $T(R, 0)$ to α. If R is an α-rigid ring (i.e., $R[x;\alpha]$ is reduced) by [6, Proposition 15], $T(R, R)$ is $\bar{\alpha}$-skew Armendariz and so it is central $\bar{\alpha}$-skew Armendariz. But $T(R, R)$ need not to be $\bar{\alpha}$-rigid. It can be asked that if $T(R, R)$ is a central $\bar{\alpha}$-skew Armendariz ring, then R is α-rigid ring. But this is not the case.

***Example* 2.12** Let $R = \mathbb{Z}_4$, where $\mathbb{Z}_4$ is the ring of integers modulo 4. Then $T(R, R)$ is a commutative ring and hence, for $\alpha = I_R$ is central $\bar{\alpha}$-skew Armendariz. But $R[x]$ is not reduced and so R is not rigid by [6, Proposition 3].

For an ideal I of R, if $\alpha(I) \subseteq I$, then $\bar{\alpha} : R/I \longrightarrow R/I$ defined by $\bar{\alpha}(a + I) = \alpha(a) + I$ is an endomorphism of a factor ring R/I. The homomorphic image of a central α-skew Armendariz ring need not be central α-skew Armendariz. But, by [6, Proposition 9], if for any $a \in R$, $a\alpha(a) \in I$ implies $a \in I$, then the factor ring R/I is $\bar{\alpha}$-skew Armendariz and so is central $\bar{\alpha}$-skew Armendariz.

Recall that a ring R is α-compatible if for each $a, b \in R$, $ab = 0$ if and only if $a\alpha(b) = 0$. Clearly, this may only happen when the endomorphism α is injective.

Theorem 2.13 Let α be an endomorphism of a ring R with $\alpha(1) = 1$, R be an α-compatible ring and I be an ideal of R with $\alpha(I) \subseteq I$. If I is reduced as a ring and R/I is central $\bar{\alpha}$-skew Armendariz ring, and for $f(x) = \sum_{i=0}^{n} a_i x^i$ and $g(x) = \sum_{j=0}^{m} b_j x^j \in R[x;\alpha] - \{0\}$, if $f(x)g(x) = 0$ and $a_0 \in U(R)$, then R is central α-skew Armendariz.

Proof. Let $a, b \in R$. Since R is α-compatible $ab = 0$ implies that for any $n \in \mathbb{N}$, $a\alpha^n(b) = 0$. Then $(\alpha^n(b)Ia)^2 = 0$. Since $\alpha^n(b)Ia \subseteq I$ and I is reduced, $\alpha^n(b)Ia = 0$. Also, $(aI\alpha^n(b))^3 \subseteq (aI\alpha^n(b))(I)(aI\alpha^n(b)) = 0$. Therefore, $aI\alpha^n(b) = 0$. Assume $f(x) = a_0 + a_1x + \cdots + a_nx^n$, $g(x) = b_0 + b_1x + \cdots + b_mx^m \in R[x;\alpha]$ and $f(x)g(x) = 0$. Then

$$a_0b_0 = 0 \tag{6}$$

$$a_0b_1 + a_1\alpha(b_0) = 0 \tag{7}$$

$$a_0b_2 + a_1\alpha(b_1) + a_2\alpha^2(b_0) = 0 \tag{8}$$

We first show that for any $a_i\alpha^i(b_j)$, $a_iI\alpha^i(b_j) = \alpha^i(b_j)Ia_i = 0$. Multiply (7) from the right by $I\alpha^n(b_0)$, we have $a_1\alpha(b_0)I\alpha^n(b_0) = 0$, since $a_0b_1I\alpha^n(b_0) = 0$. Then $(\alpha^n(b_0)Ia_1)^3 \subseteq \alpha^n(b_0)I(a_1\alpha^n(b_0)Ia_1\alpha^n(b_0))Ia_1 = 0$. Hence, $\alpha^n(b_0)Ia_1 = 0$. This implies $a_1I\alpha^n(b_0) = 0$. Multiply (7) from the left by a_0I, we have $a_0Ia_0b_1 + a_0Ia_1\alpha(b_0) = 0$ and so, $a_0Ia_0b_1 = 0$. Thus, $(b_1Ia_0)^3 = 0$ and $b_1Ia_0 = 0$. Now, multiply (8) from right by $I\alpha^n(b_0)$. Then $a_2\alpha^2(b_0)I\alpha^n(b_0) = 0$ and $(\alpha^2(b_0)Ia_2)^3 = 0$. So, $\alpha^2(b_0)Ia_2 = 0$, $a_2I\alpha^2b_0 = 0$ and $a_2I\alpha^2(b_0) = 0$. Now, from (8), we have $a_0b_2I + a_1\alpha(b_1)I + a_2\alpha^2(b_0)I = 0$. Since $a_0b_1 + a_1\alpha(b_0) = 0$ and $\alpha(a_0) = a_0$, the square of a_0b_2I and $a_2\alpha^2(b_0)I$ are zero, $a_0b_2I = a_2\alpha^2(b_0)I = 0$. Hence, $a_1\alpha(b_1)I = 0$. Then $(\alpha(b_1)Ia_1)^2 = 0$ and $\alpha(b_1)Ia_1 = 0$. So, $a_1I\alpha(b_1) = 0$. Continuing in this way, we have $a_iI\alpha^i(b_j) = \alpha^i(b_j)Ia_i = 0$. Since R/I is central α-skew Armendariz, it follows that $a_i\alpha^i(b_j) \in C(R/I)$. So, $a_i\alpha^i(b_j)r - ra_i\alpha^i(b_j) \in I$ for any $r \in R$. Now, from the above results, we have $(a_i\alpha^i(b_j)r - ra_i\alpha^i(b_j))I(a_i\alpha^i(b_j)r - ra_i\alpha^i(b_j)) = 0$. Then $a_i\alpha^i(b_j)r = ra_i\alpha^i(b_j)$ for all $r \in R$. Hence, $a_i\alpha^i(b_j)$ is central for all i and j. This completes the proof. ■

Note that in Theorem 2.13, if R is an α-rigid ring instead of α-compatible, then R should be central α-skew Armendariz by [6, Proposition 8]. The following example, shows that there exists a non-identity endomorphism α of a ring R such that R/I is central $\bar{\alpha}$-skew Armendariz and as a ring I is central α-skew Armendariz for any nonzero proper ideal I of R, but R is not central α-skew Armendariz.

***Example* 2.14** Let F be a field, $R = \left(\begin{smallmatrix} F & F \\ 0 & F \end{smallmatrix}\right)$ be a ring and $\alpha\Big(\left(\begin{smallmatrix} a & b \\ 0 & c \end{smallmatrix}\right)\Big) = \left(\begin{smallmatrix} a & -b \\ 0 & c \end{smallmatrix}\right)$ an endomorphism α of R. For $f(x) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) + \left(\begin{smallmatrix} 1 & 1 \\ 0 & 0 \end{smallmatrix}\right)x$ and $g(x) = \left(\begin{smallmatrix} 0 & 0 \\ 0 & -1 \end{smallmatrix}\right) + \left(\begin{smallmatrix} 0 & 1 \\ 0 & 1 \end{smallmatrix}\right)x \in R[x;\alpha]$, we have $f(x)g(x) = 0$, but $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 0 \end{smallmatrix}\right)\alpha\Big(\left(\begin{smallmatrix} 0 & 0 \\ 0 & -1 \end{smallmatrix}\right)\Big) = \left(\begin{smallmatrix} 0 & -1 \\ 0 & 0 \end{smallmatrix}\right) \notin C(R)$. Thus, R is not central α-skew Armendariz. But by [6, Example 12], for $I = \left(\begin{smallmatrix} F & F \\ 0 & 0 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 0 & F \\ 0 & F \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 & F \\ 0 & 0 \end{smallmatrix}\right)$, R/I and I are $\bar{\alpha}$-skew Armendariz and α-skew Armendariz, respectively. Thus, R/I and I are central $\bar{\alpha}$-skew Armendariz and central α-skew Armendariz, respectively.

Now, we have the following result.

Theorem 2.15 Let α be a monomorphism of a ring R, and $\alpha(1) = 1$ where 1 denotes the identity of R. If R is α-rigid, then a factor ring $\frac{R[x]}{\langle x^2\rangle}$ is central $\bar{\alpha}$-skew Armendariz, where $\langle x^2\rangle$ is an ideal of $R[x]$ generated by x^2. The converse holds if R is prime.

Proof. Let R be α-rigid. Then, by [6, Proposition 8], $\frac{R[x]}{\langle x^2\rangle}$ is $\bar{\alpha}$-skew Armendariz, and so is central $\bar{\alpha}$-skew Armendariz. Conversely, assume that $\frac{R[x]}{\langle x^2\rangle}$ is central α-skew Armendariz. Let $r \in R$ with $\alpha(r)r = 0$. Then

$$(\alpha(r) - \bar{x}y)(r + \bar{x}y) = \alpha(r)r + (\alpha(r)\bar{x} - \bar{x}\alpha(r))y - \alpha(1)\bar{x}^2y^2 = \bar{0},$$

Since $\alpha(r)\bar{x} = \bar{x}\alpha(r)$ in $\frac{R[x]}{\langle x^2\rangle}[y;\alpha]$, where $\bar{x} = x + \langle x^2\rangle \in \frac{R[x]}{\langle x^2\rangle}$. Since $\frac{R[x]}{\langle x^2\rangle}$ is central α-skew Armendariz, then $\alpha(r)\bar{x} \in C(\frac{R[x]}{\langle x^2\rangle})$. Thus, $\alpha(r)\bar{x}\bar{a} = \bar{a}\alpha(r)\bar{x}$ for any $a \in R$. Then $\alpha(r)a = a\alpha(r)$. Hence, $\alpha(r)Rr = 0$. Since R is prime and α is injective, $r = 0$. Therefore, R is α-rigid. ■

Let α be an automorphism of a ring R. Suppose that there exists the classical right quotient ring $Q(R)$ of R. Then for any $ab^{-1} \in Q(R)$, where $a, b \in R$ with b regular, the induced map $\bar{\alpha} : Q(R) \longrightarrow Q(R)$ defined by $\bar{\alpha}(ab^{-1}) = \alpha(a)\alpha(b)^{-1}$ is also an automorphism. Note that R is α-rigid if and only if $Q(R)$ is $\bar{\alpha}$-rigid. Hence, if R is α-rigid, then $Q(R)$ is $\bar{\alpha}$-skew Armendariz, and so is central $\bar{\alpha}$-skew Armendariz.
Let S denote a multiplicatively closed subset of a ring R consisting of central regular elements and RS^{-1} be the localization of R at S.

Proposition 2.16 Let α be an automorphism of a ring R. Then R is central α-skew Armendariz, if and only if RS^{-1} is central $\bar{\alpha}$-skew Armendariz.

Proof. Suppose that R is central α-skew Armendariz ring. Let $f(x) = \sum_{i=0}^{n}(a_i/s_i)x^i$, $g(x) = \sum_{j=0}^{m}(b_j/d_j)x^j \in RS^{-1}[x;\alpha]$ and $f(x)g(x) = 0$. Also, let $a_is_i^{-1} = c^{-1}a'_i$ and $b_jd_j^{-1} = d^{-1}b'_j$ with c, d regular elements in R. Then we have

$$(a'_0 + \cdots + a'_nx^n)d^{-1}(b'_0 + \cdots + b'_mx^m) = 0.$$

We know that for each element $f(x) \in RS^{-1}[x;\bar{\alpha}]$, there exists a regular element $c \in R$ such that $f(x) = h(x)c^{-1}$ for some $h(x) \in R[x;\alpha]$, or equivalently, $f(x)c \in R[x;\alpha]$. Therefore, there exist a regular element e in R and $(b''_0 + \cdots + b''_tx^t) \in R[x;\alpha]$ such that $d^{-1}(b'_0 + \cdots + b'_mx^m) = (b''_0 + \cdots + b''_tx^t)e^{-1}$. Hence,

$$(a'_0 + \cdots + a'_nx^n)(b''_0 + \cdots + b''_tx^t) = 0.$$

Since R is central α-skew Armendariz, $a'_i\alpha^i(b''_j) \in C(R)$ for all i and j. Therefore, $ca_is_i^{-1}\alpha^i(b_jd_j^{-1}e^{-1}) \in C(R)$. Since c and e are regular element of R, $a_is_i^{-1}\alpha(b_jd_j^{-1})$ are central in R for all i and j. Conversely, assume that RS^{-1} is central α-skew Armendariz ring. Since subrings of central α-skew Armendariz rings are central α-skew Armendariz, then R is central α-skew Armendariz. ■

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