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A Nonlocal First Order Shear Deformation Theory for Vibration Analysis of Size Dependent Functionally Graded Nano beam with Attached Tip Mass: an Exact Solution

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ABSTRACT

In this article, transverse vibration of a cantilever nano- beam with functionally graded materials and carrying a concentrated mass at the free end is studied. Material properties of FG beam are supposed to vary through thickness direction of the constituents according to power-law distribution (P-FGM). The small scale effect is taken into consideration based on nonlocal elasticity theory of Eringen. The nonlocal equations of motion are derived based on Timoshenko beam theory in order to consider the effect of shear deformation and rotary inertia. Hamilton's principle is applied to obtain the governing differential equation of motion and boundary conditions and they are solved applying analytical solution. The purpose is to study the effects of parameters such as tip mass, small scale, beam thickness, power-law exponent and slenderness on the natural frequencies of FG cantilever nano beam with a point mass at the free end. It is explicitly shown that the vibration behavior of a FG Nano beam is significantly influenced by these effects. The response of Timoshenko Nano beams obtained using an exact solution in a special case is compared with those obtained in the literature and is found to be in good agreement. Numerical results are presented to serve as benchmarks for future analyses of FGM cantilever Nano beams with tip mass.

Keywords: Timoshenko beam theory; Free vibration; Functionally graded Nano beam; Nonlocal elasticity theory; Tip mass.

1 INTRODUCTION

FUNCTIONALLY graded material are new type of composite materials formed of two or multi phases which both it's composition and structure gradually change over gradient directions smoothly and continuously, therefore by changing the properties of the material it is possible to perform a certain function of material properties of mechanical strength and thermal conductivity. These materials which is introduced by Japanese scientists in mid-1980s possess various advantage in comparison with traditional composites, for instance, multifunctionality, ability to control deformation, corrosion and dynamic response, minimizations or remove stress concentrations, smoothing the transition of thermal stress, resistance to oxidation. Hence FGMs have received wide engineering applications in modern industries including aerospace, nuclear energy applications, turbine components, rocket nozzles, chemical

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reactor tubes, batteries/fuel cells, critical furnace parts, etc. during the past two decades. These wide engineering applications is cause that researchers attracted to FGMs, and study their vibration, static and dynamic's behavior of the FG structures. Many investigation are reported in literature to study the dynamic and static behavior of functionally graded beams, here some of these disquisitions are mentioned briefly.

Moreover, structural elements such as beams, plates, and membranes in micro or Nano length scale are commonly used as components in micro/nano electromechanical systems (MEMS/NEMS). Therefore understanding the mechanical and physical properties of nanostructures is necessary for its practical applications. Nanoscale engineering materials have attracted great interest in modern science and technology after the invention of carbon nanotubes (CNTs) by Iijima, (1991) [1]. They have significant mechanical, thermal and electrical performances that are superior to the conventional structural materials. In recent years, Nano beams and CNTs hold a wide variety of potential applications [2] such as sensors, actuators, transistors, probes, and resonators in NEMSs. For instance, in MEMS/NEMS; nanostructures have been used in many areas including communications, machinery, information technology, biotechnology technologies. Since conducting experiments at the nanoscale is a daunting task, and atomistic modeling is restricted to small-scale systems owing to computer resource limitations, continuum mechanics offers an easy and useful tool for the analysis of CNTs. However the classical continuum models need to be extended to consider the nanoscale effects and this can be achieved through the nonlocal elasticity theory proposed by Eringen [3] which consider the size-dependent effect. According to this theory, the stress state at a reference point is considered as a function of strain states of all points in the body. This nonlocal theory is proved to be in accordance with atomic model of lattice dynamics and with experimental observations on phonon dispersion [4]. Moreover, in recent years the application of nonlocal elasticity theory, in micro and nanomaterials has received a considerable attention within the nanotechnology community-. Peddieson et al. [5] proposed a version of nonlocal elasticity theory which is employed to develop a nonlocal Bernoulli/Euler beam model. Wang and Liew [6] carried out the static analysis of micro- and nano-structures based on nonlocal continuum mechanics using Euler-Bernoulli beam theory and Timoshenko beam theory. Aydogdu [6] proposed a generalized nonlocal beam theory to study bending, buckling, and free vibration of Nano beams based on Eringen model using different beam theories. Phadikar and Pradhan [7] reported finite element formulations for nonlocal elastic Euler–Bernoulli beam and Kirchhoff plate. Pradhan and Murmu [8] investigated the flap wise bending–vibration characteristics of a rotating Nano cantilever by using Differential quadrature method (DQM). They noticed that small-scale effects play a significant role in the vibration response of a rotating Nano cantilever. Ghorbanpour et al. [9] researched wave propagation of Nano beams resting on a Pasternak Foundation with considering surface stress effect. Ansari et al.[10] employed modified couple stress theory for investigating the vibration characteristic of a post-bucked micro beam based on Euler-Bernoulli theory with different boundary conditions, they considered the geometric nonlinearity by using Von Karman strain tensor. They find out that the stiffness of the beam was increased by considering of the larger value of the material length-scale parameter to the thickness ratio. Thermal effect on free vibration behavior of FG Nano beams based on Euler-Bernoulli with three type of boundary conditions was investigated by Ebrahimi and salari [11], the small scale effect was based on nonlocal elasticity theory of Eringen, materials properties was assumed to be the temperature-dependent. They used differential transform method and analytical solution based on Navier type method and compared the result of two methods, also concluded that small scale has an important effect on vibrations of Nano beams. Determination of the influence of the parameters characterizing a system on the vibration of the system is of practical interest in engineering applications. Many factors can affect the flexural vibration of beams, in particular the axial load, intermediate supports and attached masses. Free vibrations of a beam having concentrated masses are extensively studied. Beam–mass systems are frequently used as design models in engineering.

Studies on the vibration response of FG nano structures with attached tip mass, especially for beams, are still limited in number. for instance. Exact and approximate analyses have been carried out for calculating the natural frequencies of a beam–mass system under simple supported condition [11–20]. There are many studies about free vibration of beams with different boundary conditions and about free vibration of beams with any number of attachments in this study [21–25]. Most of these studies were presented without considering the effects of Timoshenko beam theory. They utilized Euler–Bernoulli beam theory while Timoshenko beam theory has more accurate results than Euler–Bernoulli theory, and the number of studies using this theory is limited. Considering the influence of masses on a shaft or beam is very important due to the decrease of natural frequencies of the shaft or beam in the presence of concentrated masses. This reduction should be considered in designing and manufacturing of structures, shafts and other applications. As mentioned above, some researchers were studied the vibration of the beams with concentrated masses by Euler–Bernoulli theory. Laura et al. [26] studied an Euler–Bernoulli Cantilever beam with a point mass and. He studied only one boundary condition and considered one position for a concentrated mass. The transverse vibration of a beam with an arbitrary placed concentrated mass and elastically restrained-hinged boundary condition at both ends was conducted by Goel [27]. He used Dirac's delta to impose the effect of one concentrated mass to governing equation and used Laplace transform in his solution. Parnell and Cobble [28] studied lateral displacement of a vibrating Cantilever beam with a concentrated mass with general boundary condition by Laplace transform method. They also considered one position for the point mass. A research on vibration of a Cantilever beam with a concentrated mass and base excitation was carried out by To [29]. He imposed the effect of distance between tip mass center of gravity and point of its attachment to end of the beam. Grant [30] employed Timoshenko beam theory for obtaining the frequency and normal mode of uniform beams carrying a concentrated mass. He used Dirac's delta function to represent the effects of the concentrated mass on Timoshenko beam and then solved governing equations by Laplace transform method. Bruch and Mitchell [31] studied vibration of a Clamped-Free Timoshenko beam which carries lumped mass-rotary inertia on its free end. He proved the reduction of first five natural frequencies of beam due to increasing mass or rotating inertia of lumped mass-rotary inertia. Abramovich and Hamburger [32] considered the effect of distance between the tip mass centroid and the point of tip mass attachment on the transverse vibration of a Cantilever beam carrying a tip mass at its free end. This effect causes a moment at the end of beam, and accompanied by effects of Timoshenko beam theory in the vibration of beam with tip mass. He compared the obtained results with results of Bruch [31]. In another research [33] Abramovich and Hamburger restudied vibration of a uniform Cantilever Timoshenko beam with translational and rotational springs and with a tip mass. In Ref. [34] Rossi et al. investigated free vibrations of Timoshenko beams carrying elastically mounted concentrated masses. He used governing equations of Timoshenko beam, and then compatibility conditions were used to impose the effect of shear force which is caused by mass-spring system on transverse vibration of a Timoshenko beam. In recent years, scientists tried to solve more complex problems related to the effect of concentrated mass on vibration of beams. Salarieh and Ghrashi [35] studied the effect of finite mass on both torsional and transverse vibration of Timoshenko beam. Free vibration analyses of an immersed beam carrying an eccentric tip mass with rotary inertia is performed by Wu and Hsu [36]. Lin and Tsai [37] used Euler–Bernoulli beam theory to analyze a uniform multi-span beam carrying multiple spring-mass systems. They only used Pinned–Pinned boundary condition for the concerned beam. Togun [38] provided multiple scale method for nonlinear analyze of free and forced vibration of nano beam with attached mass based on Euler–Bernoulli beam theory. Finally some researchers used numerical procedures to investigate free vibrations of non-uniform beams carrying concentrated mass or masses.

As seen, there is no study investigating the tip mass effect on vibration behavior of functionally graded Nano beams based on Timoshenko beam theory, while there is strong scientific need to understand the vibration behavior of FG nano beam with tip mass. It is assumed that material properties of the beam, vary continuously through the beam thickness according to power-law form. Governing equations and boundary conditions for the free vibration of a FG nano beam have been derived via Hamilton's principle. An exact method is employed to solve governing equations for vibration behavior analysis of FG nano beams with tip mass.

2 THEORY AND FORMULATION

2.1 Power-law functionally graded beams with concentrated mass

Consider a uniform FG beam with tip mass and rectangular cross-section of length L, width b and thickness h according to Fig. 1. *X–axis* is matched with neutral axis of the beam in the undeflected position, the *y*–axis in the width direction, and the *z*–axis in the thickness direction. The beam is made of inhomogeneous and isotropic functionally graded materials which the volume fraction and nano-structural morphology of the material compositions are varying continuously in the thickness direction only. Functionally graded materials are the new generation of composite materials which are usually produced from two or multi different materials. In this study FG material is made from a mixture of ceramic and metal and the material properties of FG beam are supposed to vary through thickness direction of the constitutes according to power-low distribution. The effective material properties of FG beam that distributed identical in two phases of ceramic and metal can be expressed by using the rule of mixture as:

$$p_f = p_c v_c + p_m v_m \tag{1}$$

where p_c and p_m are the material properties of ceramic and metal v_c and v_m are the volume fraction of ceramic and metal that are attached as:

$$v_c + v_m = 1 \tag{2a}$$

The power-low volume fraction of the ceramics constituents of the beam is assumed to be given by;

$$v_c = (\frac{z}{h} + \frac{1}{2})^P \tag{2b}$$

Here z is the distance from the mid-plane of the FGM beam and p is the non-negative variable parameter (power-low exponent) which determines the material distribution through the thickness of the beam. According to this distribution we have a fully metal beam for large value of p and when p equal to zero a fully ceramic beam remain. Effective material properties such as Young's modulus (E), Poisson's ratio (υ) and mass density (ρ) are assumed to vary continuously in the thickness direction according to power-law distribution. P-FGM is one of the most favorable models for FGMs. The effective material properties of FG beam can be expressed by:

$$E(z) = (E_c - E_m)\left(\frac{z}{h} + \frac{1}{2}\right)^p + E_m \tag{3a}$$

$$\rho(z) = (\rho_c - \rho_m)\left(\frac{z}{h} + \frac{1}{2}\right)^p + \rho_m \tag{3b}$$

$$\upsilon(z) = (\upsilon_c - \upsilon_m)\left(\frac{z}{h} + \frac{1}{2}\right)^p + \upsilon_m \tag{3c}$$

Fig.1
Geometry and coordinates of functionally graded cantilever Nano beam attached to a concentrated mass at the free end.

2.2 Kinematic relations

The equations of motion are derived based on the Timoshenko beam theory, so that the displacement field at any point of the beam can be written as:

$$u_x(x,z,t) = u(x,t) + z\,\varphi(x,t) \tag{4a}$$

$$u_z(x,z,t) = w(x,t) \tag{4b}$$

where u is the axial displacement along x–axis, w is the transverse displacement along z–axis, φ is the rotational angle due to bending and t is the time. Then the strains field can be expressed as:

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} + z\frac{\partial \varphi}{\partial x} \tag{5a}$$

$$\gamma_{xz} = \frac{\partial w}{\partial x} + \varphi \tag{5b}$$

ε_{xx} , γ_{xz} are normal and shear strain. The Euler Lagrange equations has been used to derive the equation of motion by using a Hamilton's principle, which` can be stated as:

$$\int_{t_1}^{t_2} \delta(U - T + V)\, dt = 0 \tag{6}$$

where t_1, t_2 are the initial and end time δU is the virtual variation of strain energy, δV is the virtual variation of work done by external loads, δT is the virtual variation of kinetic energy. Here strain energy, kinetic energy and potential energy (external loading) can be calculated step by step and the equations of motion has been obtained by using rules of calculus of variations and Hamilton's principle.

In the first step we define strain energy as:

$$\delta U = \int_v \sigma_{ij}\, \delta\varepsilon_{ij}\, dV = \int_0^x \int_A (\sigma_{xx}\, \delta\varepsilon_{xx} + \sigma_{xz}\, \delta\gamma_{xz})\, dA\, dx \tag{7}$$

where δ is the variational symbol, A is the cross-section area of the uniform beam, σ_{xx} the axial stress and σ_{xz} is the shear stress, by substituting the expressions for $\varepsilon_{xx}, \gamma_{xz}$ into Eq. (7) as:

$$\delta U = \int_0^L \int_A \sigma_{xx}\, \delta(\frac{\partial u}{\partial x} + z\, \frac{\partial \varphi}{\partial x}) dA\, dx + \int_0^L \int_A \sigma_{xz}\, \delta(\frac{\partial w}{\partial x} + \varphi) dA\, dx \tag{8}$$

By defining N, M, Q as axial force, bending moment and shear force components as following:

$$(N, M) = \int_A \sigma_{xx}\, (1, z)\, dA \tag{9}$$

$$Q = \int_A K_s \sigma_{xz}\, dA \tag{10}$$

where coefficient K_s is called the Timoshenko shear correction factor. The exact value of K_s is function of material properties and cross section parameters of the beam. Here for rectangular beams K_s has been assumed is equal to $5/6$ approximately. By replacing these resultants into Eq. (8), get to:

$$\delta U = \int_0^L (N\, (\frac{\partial \delta u}{\partial x}) + M\, (\frac{\partial \delta\varphi}{\partial x}) + Q\, (\frac{\partial \delta w}{\partial x} + \delta\varphi))\, dx \tag{11}$$

$$\delta U = N\, \delta u \,\Big]_{x=0}^{x=L} - \int_0^L \frac{\partial N}{\partial x} \delta u + M\, \delta\varphi \,\Big]_{x=0}^{x=L} - \int_0^L \frac{\partial M}{\partial x} \delta\varphi + Q\, \delta w \,\Big]_{x=0}^{x=L} - \int_0^L \frac{\partial Q}{\partial x} \delta w + \int_0^L Q\, \delta\varphi \tag{12}$$

$$\delta U = \left[N\, \delta u + M\, \delta\varphi + Q\, \delta w\,\right]_{x=0}^{x=L} - \int_0^L \left[\frac{\partial N}{\partial x}\, \delta u + \frac{\partial M}{\partial x}\, \delta\varphi + \frac{\partial Q}{\partial x}\, \delta w + Q\, \delta\varphi\right] \tag{13}$$

In the second step, the kinetic energy expression for Timoshenko beam with concentrated mass can be expressed as:

$$T_{Total} = T_{Beam} + T_{Tip\ mass} \tag{14}$$

$$T_{Tipmass} = \frac{1}{2} m \left[\left(\frac{\partial u_x}{\partial t} \right)^2 + \left(\frac{\partial u_y}{\partial t} \right)^2 + \left(\frac{\partial u_z}{\partial t} \right)^2 \right]_{x=x_m} = \frac{1}{2} m \left[\frac{\partial u}{\partial t}^2 + \frac{\partial w}{\partial t}^2 \right]_{x=x_M=L} \tag{15}$$

$$T_{Beam} = \frac{1}{2}\int_0^L \int_A \rho(z)\left((\frac{\partial u_x}{\partial t})^2 + (\frac{\partial u_y}{\partial t})^2 + (\frac{\partial u_z}{\partial t})^2 \right) dA\, dx \tag{16}$$

$$T_{Beam} = \frac{1}{2}\int_0^L \int_A \rho(z)[(\frac{\partial u}{\partial t})^2 + z^2(\frac{\partial \varphi}{\partial t})^2 + 2z\frac{\partial \varphi}{\partial t}\frac{\partial u}{\partial t} + (\frac{\partial w}{\partial t})^2]\,\mathrm{dA}dx \tag{17}$$

$$T_{Beam} = \frac{1}{2}\int_0^L I_0(\frac{\partial u}{\partial t})^2 + I_2(\frac{\partial \varphi}{\partial t})^2 + 2I_1\frac{\partial \varphi}{\partial t}\frac{\partial u}{\partial t} + I_0(\frac{\partial w}{\partial t})^2 dx \tag{18}$$

where (I_0, I_1, I_2) are the mass moment of inertias that can be defined as:

$$(I_0, I_1, I_2) = \int_A \rho(z)(1, z, z^2)\, dA \tag{19}$$

The first variation of the virtual kinetic energy can be written in the form:

$$\delta T_{Beam} = \int_0^L \left[I_0(\frac{\partial u}{\partial t}\frac{\partial \delta u}{\partial t} + \frac{\partial w}{\partial t}\frac{\partial \delta w}{\partial t}) + I_1(\frac{\partial \varphi}{\partial t}\frac{\partial \delta u}{\partial t} + \frac{\partial u}{\partial t}\frac{\partial \delta \varphi}{\partial t}) + I_2\frac{\partial \varphi}{\partial t}\frac{\partial \delta \varphi}{\partial t} \right] dx \tag{20}$$

$$\int_{t_1}^{t_2} \delta T_{Tip\ mass} = m\left[\frac{\partial^2 u}{\partial t^2}\delta u + \frac{\partial^2 w}{\partial t^2}\delta w \right]_{x=L} \tag{21}$$

In the last step, the variation of potential energy can be obtained as:

$$\delta V = \int_0^L [f(x)\delta u + q(x)\delta w + \bar{N}\frac{\partial w}{\partial x}\frac{\partial(\delta w)}{\partial x}]\mathrm{dx} = 0 \tag{22}$$

At last by substituting Eqs. (13), (20), (21) and (22) into Eq. (6) as:

$$\int_{t_1}^{t_2}\int_0^l \left(-\frac{\partial N}{\partial x} + I_0\frac{\partial^2 u}{\partial t^2} + I_1\frac{\partial^2 \varphi}{\partial t^2} \right)\delta u + \left(Q - \frac{\partial M}{\partial X} + I_1\frac{\partial^2 u}{\partial t^2} + I_2\frac{\partial^2 \varphi}{\partial t^2} \right)\delta\varphi + \left(-\frac{\partial Q}{\partial x} + I_0\frac{\partial^2 w}{\partial t^2} \right)\delta w = 0 \tag{23}$$

$$\left[N\,\delta u + M\,\delta\varphi + Q\,\delta w \right]_{x=0}^{x=L} + m[\frac{\partial^2 u}{\partial t^2}\delta u + \frac{\partial^2 w}{\partial t^2}\delta w]_{x=L} = 0 \tag{24}$$

And setting the coefficients of ∂u , $\delta\varphi$ and δw equal to zero, the governing equations of motion of FG Timoshenko beam attached tip mass can be obtained as:

$$(\delta u:0),\ \frac{\partial N}{\partial x} - I_0\frac{\partial^2 u}{\partial t^2} - I_1\frac{\partial^2 \varphi}{\partial t^2} = 0 \tag{25}$$

$$(\delta\varphi:0),\ \frac{\partial M}{\partial x} - Q - I_1\frac{\partial^2 u}{\partial t^2} - I_2\frac{\partial^2 \varphi}{\partial t^2} = 0 \tag{26}$$

$$\left(\delta w\ :0\right),\ \frac{\partial Q}{\partial x}-I_0\frac{\partial^2 w}{\partial t^2}=0 \tag{27}$$

2.3 The nonlocal elasticity model for FG Nano beam based on Timoshenko beam theory

Based on Eringen nonlocal elasticity model, the stress at a reference point x in a body is considered as a function of strains of all points in the near region. This assumption is agreement with experimental observations of atomic theory and lattice dynamics in phonon scattering in which for a nonhomogeneous and isotropic elastic solid; the nonlocal stress-tensor components σ_{ij} at any point x in the body can be expressed as:

$$\sigma_{ij}(x)=\int_{\Omega}\alpha(|x'-x|,\tau)\,t_{ij}(x')\,d\Omega \tag{28}$$

$$t_{ij}=C_{ijkl}\varepsilon_{kl} \tag{29}$$

$$\tau=\frac{e_0 a}{l} \tag{30}$$

The magnitude of e_0 is determined experimentally or approximated by matching the dispersion curves of plane waves with those of atomic lattice dynamics. According to nonlocal theory for a class of physically admissible kernel $\alpha(|x'-x|,\tau)$ it is possible to represent the integral constitutive relations given by Eq. (28) in an equivalent differential form as:

$$(1-(e_0 a)^2\nabla^2)\sigma_{kl}=t_{kl} \tag{31}$$

$$\sigma_{xx}-(e_0 a)^2\frac{\partial^2\sigma_{xx}}{\partial x^2}=E\,\varepsilon_{xx} \tag{32}$$

$$\sigma_{xz}-(e_0 a)^2\frac{\partial^2\sigma_{xz}}{\partial x^2}=G\gamma_{xz} \tag{33}$$

where σ and ε are the nonlocal stress and strain, respectively. E is the Young's modulus, $G=E(z)/2(1+\nu(z))$ is the shear modulus (where v is the Poisson's ratio). By defining $(A_{xx}, B_{xx}, D_{xx}, C_{xz})$ and replacing these resultants into Eqs. (32) and (33) and integrating over the beam's cross-section area, the force-strain and the moment-strain of the nonlocal Timoshenko FG beam theory can be obtained as following:

$$N-\mu\frac{\partial^2 N}{\partial x^2}=A_{xx}\frac{\partial u}{\partial x}+B_{xx}\frac{\partial\varphi}{\partial x} \tag{34}$$

$$M-\mu\frac{\partial^2 M}{\partial x^2}=B_{xx}\frac{\partial u}{\partial x}+D_{xx}\frac{\partial\varphi}{\partial x} \tag{35}$$

$$Q-\mu\frac{\partial^2 Q}{\partial x^2}=C_{xz}\left(\frac{\partial w}{\partial x}+\varphi\right) \tag{36}$$

In which the cross-section stiffness are defined as:

$$(A_{xx}, B_{xx}, D_{xx}) = \int_A E(z)(1, z, z^2)\, dA \tag{37}$$

$$C_{xz} = K_s \int_A G(z)\; dA \tag{38}$$

The explicit relation of the nonlocal normal force can be derived by substituting for the second derivative of *N* from Eq. (25) into Eq. (34) as follows:

$$N = A_{xx} \frac{\partial u}{\partial x} + B_{xx} \frac{\partial \varphi}{\partial x} + \mu \left(I_0 \frac{\partial^3 u}{\partial x\, \partial t^2} + I_1 \frac{\partial^3 \varphi}{\partial x\, \partial t^2} \right) \tag{39}$$

Also, the explicit relation of the nonlocal bending moment can be derived by substituting for the second derivative of *M* from Eq. (26) into Eq. (35) as follows:

$$M = B_{xx} \frac{\partial u}{\partial x} + D_{xx} \frac{\partial \varphi}{\partial x} + \mu \left(I_0 \frac{\partial^2 w}{\partial t^2} + I_1 \frac{\partial^3 u}{\partial x\, \partial t^2} + I_2 \frac{\partial^3 \varphi}{\partial x\, \partial t^2} \right) \tag{40}$$

By substituting for the second derivative of *Q* from Eq. (27) into Eq. (36), the following expression for the nonlocal shear force is derived:

$$Q = C_{xz} \left(\frac{\partial w}{\partial x} + \varphi\right) + \mu \left(I_0 \frac{\partial^3 w}{\partial x\, \partial t^2} \right) \tag{41}$$

The nonlocal governing equations of Timoshenko FG Nano beam in terms of the displacement can be derived by substituting for *N*, *M* and *Q* from Eqs. (39) - (41), respectively, into Eq. (34) - (36) as follows:

$$A_{\mathrm{xx}} \frac{\partial^2 u}{\partial x^2} + B_{\mathrm{xx}} \frac{\partial^2 \varphi}{\partial x^2} + \mu \left(I_0 \frac{\partial^4 u}{\partial t^2 \partial x^2} + I_1 \frac{\partial^4 \varphi}{\partial t^2 \partial x^2} \right) - I_0 \frac{\partial^2 u}{\partial t^2} - I_1 \frac{\partial^2 \varphi}{\partial t^2} = 0 \tag{42}$$

$$C_{xz} \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial \varphi}{\partial x}\right) + \mu \left(I_0 \frac{\partial^4 w}{\partial t^2 \partial x^2}\right) - I_0 \frac{\partial^2 w}{\partial t^2} = 0 \tag{43}$$

$$B_{\mathrm{xx}} \frac{\partial^2 u}{\partial x^2} + D_{\mathrm{xx}} \frac{\partial^2 \varphi}{\partial x^2} - C_{xz} \left(\frac{\partial w}{\partial x} + \varphi\right) + \mu \left(I_1 \frac{\partial^4 u}{\partial t^2 \partial x^2} + I_2 \frac{\partial^4 \varphi}{\partial t^2 \partial x^2}\right) - I_1 \frac{\partial^2 u}{\partial t^2} - I_2 \frac{\partial^2 \varphi}{\partial t^2} = 0 \tag{44}$$

By ignoring the axial displacement, the nonlocal governing equations of Timoshenko FG Nano beam and boundary condition can be obtained as:

$$C_{xz} \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial \varphi}{\partial x}\right) + \mu \left(I_0 \frac{\partial^4 w}{\partial t^2 \partial x^2}\right) - I_0 \frac{\partial^2 w}{\partial t^2} = 0 \tag{45}$$

$$D_{\mathrm{xx}} \frac{\partial^2 \varphi}{\partial x^2} - C_{xz} \left(\frac{\partial w}{\partial x} + \varphi\right) + \mu \left(I_2 \frac{\partial^4 \varphi}{\partial t^2 \partial x^2}\right) - I_2 \frac{\partial^2 \varphi}{\partial t^2} = 0 \tag{46}$$

By considering clamped-free, the boundary condition can be stated as following;

$$\left[D_{xx}\frac{\partial\varphi}{\partial x}+\mu\left(I_0\frac{\partial^2 w}{\partial t^2}+I_2\frac{\partial^3\varphi}{\partial x\,\partial t^2}\right)\right]_{x=L}=0 \qquad ,\left[C_{xz}\left(\frac{\partial w}{\partial x}+\varphi\right)+\mu\left(I_0\frac{\partial^3 w}{\partial x\,\partial t^2}\right)+m\frac{\partial^2 w}{\partial t^2}\right]_{x=L}=0 \tag{47}$$

$$\delta w=0\big|_{x=0}\ \textit{and}\ \ \delta\varphi=0\big|_{x=0} \tag{48}$$

3 SOLUTION METHOD: ANALYTICAL SOLUTION

Assuming a sinusoidal variation of $w(x,t)$ and $\varphi(x,t)$, which the functions are approximated as:

$$w(x,t)=\bar{w}e^{i\omega t} \tag{49}$$

and

$$\varphi(x,t)=\bar{\varphi}e^{i\omega t} \tag{50}$$

And by substituting Eq. (50) into Eqs. (45), (46), (48) and (49), equations of motion and boundary conditions are obtained as:

$$C_{xz}\left(\frac{\partial^2\bar{w}}{\partial x^2}+\frac{\partial\bar{\varphi}}{\partial x}\right)+\mu\omega^2\left(I_0\frac{\partial^2\bar{w}}{\partial x^2}\right)-I_0\omega^2\bar{w}=0 \tag{51}$$

$$D_{xx}\frac{\partial^2\bar{\varphi}}{\partial x^2}-C_{xz}\left(\frac{\partial\bar{w}}{\partial x}+\bar{\varphi}\right)+\mu\omega^2\left(I_2\frac{\partial^2\bar{\varphi}}{\partial x^2}\right)-I_2\omega^2\bar{\varphi}=0 \tag{52}$$

Boundary conditions:

$$\left[D_{xx}\frac{\partial\bar{\varphi}}{\partial x}+\mu\omega^2\left(I_0\bar{w}+I_2\frac{\partial\bar{\varphi}}{\partial x}\right)\right]_{x=L}=0 \qquad ,\left[+C_{xz}\left(\frac{\partial\bar{w}}{\partial x}+\bar{\varphi}\right)+\mu I_0\omega^2\frac{\partial\bar{w}}{\partial x}+m\,\omega^2\bar{w}\right]_{x=L}=0 \tag{53}$$

$$\delta\bar{w}=0_{\ x=0} \qquad \delta\bar{\varphi}=0_{\ x=0} \tag{54}$$

By finding first derivation of $\bar{\varphi}$ and $\bar{w}$ from Eqs. (51) and (52) and calculating the second and third derivations of them. Then by deriving over the Eqs. (51) and (52), and substituting the first and third derivation of $\bar{\varphi}$ and $\bar{w}$ yield:

$$A_{11}\frac{\partial^4\bar{\varphi}}{\partial x^4}+B_{11}\frac{\partial^2\bar{\varphi}}{\partial x^2}+C_{11}\bar{\varphi}=0 \tag{55}$$

$$A_{11}\frac{\partial^4\bar{w}}{\partial x^4}+B_{11}\frac{\partial^2\bar{w}}{\partial x^2}+C_{11}\bar{w}=0 \tag{56}$$

$$\left[D_{xx}\frac{\partial\bar{\varphi}}{\partial x}+\mu\omega^2\left(I_0\bar{w}+I_2\frac{\partial\bar{\varphi}}{\partial x}\right)\right]_{x=L}=0 \qquad ,\left[C_{xz}\left(\frac{\partial\bar{w}}{\partial x}+\bar{\varphi}\right)+\mu I_0\omega^2\frac{\partial w}{\partial x}+m\,\omega^2\bar{w}\right]_{x=L}=0 \tag{57}$$

$$\delta\bar{w} = 0 \Big|_{x=0} \qquad \delta\bar{\varphi} = 0 \Big|_{x=0} \tag{58}$$

where

$$A_{11} = \left[\frac{[D_{xx} + \mu\omega^2 I_2][C_{xz} + \mu\omega^2 I_0]}{C_{xz}}\right],\ B_{11} = \left[C_{xz} + \frac{-I_0\omega^2[D_{xx} + \mu\omega^2 I_2] - [C_{xz} + I_2\omega^2][C_{xz} + \mu\omega^2 I_0]}{C_{xz}}\right],$$
$$C_{11} = \frac{I_0\omega^2[C_{xz} + I_2\omega^2]}{C_{xz}} \tag{59}$$

Solution to Eqs. (55) and (56) are easily found to be as follow:

$$\bar{w} = c_1\cosh(\alpha x) + c_2\sinh(\alpha x) + c_3\cos(\beta x) + c_4\sin(\beta x)$$
$$\bar{\varphi} = c_1'\sinh(\alpha x) + c_2'\cosh(\alpha x) + c_3'\sin(\beta x) + c_4'\cos(\beta x) \tag{60}$$

In which

$$(\alpha,\beta) = \pm\sqrt{\frac{-B_{11} + \sqrt{B_{11}{}^2 - 4A_{11}C_{11}}}{2A_{11}}} \tag{61}$$

The values of the constants in Eq. (62) are related by the coupling Eqs. (51) and (52) as following:

$$\frac{I_0\omega^2 - \alpha^2[C_{xz} + \mu\omega^2 I_0]}{\alpha C_{xz}}c_1 = c_1' \qquad \frac{I_0\omega^2 + \beta^2[C_{xz} + \mu\omega^2 I_0]}{\beta C_{xz}}c_3 = c_3'$$
$$\frac{I_0\omega^2 - \alpha^2[C_{xz} + \mu\omega^2 I_0]}{\alpha C_{xz}}c_2 = c_2' \qquad \frac{I_0\omega^2 + \beta^2[C_{xz} + \mu\omega^2 I_0]}{-\beta C_{xz}}c_4 = c_4' \tag{62}$$

By inserting the Eq. (61) into boundary conditions as following:

$$c_1 + c_3 = 0$$
$$\frac{I_0\omega^2 - \alpha^2[C_{xz} + \mu\omega^2 I_0]}{\alpha C_{xz}}c_2 + \frac{I_0\omega^2 + \beta^2[C_{xz} + \mu\omega^2 I_0]}{-\beta C_{xz}}c_4 = 0$$
$$\left[[\frac{I_0\omega^2 - \alpha^2[C_{xz} + \mu\omega^2 I_0]}{\alpha C_{xz}}][C_{xz}\ \mathrm{Sinh}(\alpha L)] + \alpha[C_{xz} + \mu\omega^2 I_0]\mathrm{Sinh}(\alpha L) + m\omega^2\ \mathrm{Cosh}(\alpha L)\right]c_1 +$$
$$\left[[\frac{I_0\omega^2 - \alpha^2[C_{xz} + \mu\omega^2 I_0]}{\alpha C_{xz}}][C_{xz}\ \mathrm{Cosh}(\alpha L)] + \alpha[C_{xz} + \mu\omega^2 I_0]\mathrm{Cosh}(\alpha L) + m\omega^2\ \mathrm{Sinh}(\alpha L)\right]c_2 +$$
$$\left[[\frac{I_0\omega^2 + \beta^2[C_{xz} + \mu\omega^2 I_0]}{\beta C_{xz}}][C_{xz}\ \mathrm{Sin}(\beta L)] - \beta[C_{xz} + \mu\omega^2 I_0]\mathrm{Sin}(\beta L) + m\omega^2\ \mathrm{Cos}(\beta L)\right]c_3 + \tag{63}$$
$$\left[[\frac{I_0\omega^2 + \beta^2[C_{xz} + \mu\omega^2 I_0]}{-\beta C_{xz}}][C_{xz}\ \mathrm{Cos}(\beta L)] + \beta[C_{xz} + \mu\omega^2 I_0]\mathrm{Cos}(\beta L) + m\omega^2\ \mathrm{Sin}(\beta L)\right]c_4 = 0$$
$$\left[[D_{xx} + \mu\omega^2 I_2][\frac{I_0\omega^2 - \alpha^2[C_{xz} + \mu\omega^2 I_0]}{\alpha C_{xz}}]\alpha\ \mathrm{Cosh}(\alpha L) + \mu I_0\omega^2\ \mathrm{Cosh}(\alpha L)\right]c_1 +$$

$$\left[[D_{xx}+\mu\omega^2 I_2][\frac{I_0\omega^2-\alpha^2[C_{xz}+\mu\omega^2 I_0]}{\alpha C_{xz}}]\alpha\,\mathrm{Sinh}(\alpha L)+\mu I_0\omega^2\,\mathrm{Sinh}(\alpha L)\right]c_2+$$
$$\left[[D_{xx}+\mu\omega^2 I_2]\frac{I_0\omega^2+\beta^2[C_{xz}+\mu\omega^2 I_0]}{\beta C_{xz}}\beta\,\mathrm{Cos}(\beta L)+\mu I_0\omega^2\,\mathrm{Cos}(\beta L)\right]c_3+$$
$$\left[[D_{xx}+\mu\omega^2 I_2]\frac{I_0\omega^2+\beta^2[C_{xz}+\mu\omega^2 I_0]}{\beta C_{xz}}\beta\,\mathrm{Sin}(\beta L)+\mu I_0\omega^2\,\mathrm{Sin}(\beta L)\right]c_4=0$$

The matrix equation can be expressed as follows:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}\begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{64}$$

The coefficients a_{11}, a_{12} and etc., of the matrix equation can be obtained and by taking the determinant of the coefficient matrix in Eq. (65) and setting this multinomial to zero, we can find natural frequencies ω_n.

$$det\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}=0 \tag{65}$$

The non-dimensional natural frequencies (λ) can be calculated as following:

$$\lambda=\omega\frac{L^2}{h}\sqrt{\frac{\rho_m}{e_m}} \tag{66}$$

4 NUMERICAL RESULT AND DISCUSSIONS

The functionally graded beam is combined of Aluminum (Al) and Alumina (Al_2o_3) where their properties are given in Table 1. In Table 2. Numerical results are compared with Simsek [38] and Pradhan& Chakraverty [39] for validating the present research, hereupon natural frequencies of FG beams composed of alumina and aluminum for two different values of slenderness ratio, $(L/h=5,\ 20)$ and various gradient indexes with Clamped-Free boundary condition are obtained by numerical solution method. The present frequencies are in good agreement with results of Simsek [38] and Pradhan & Chakraverty[39].

Table 1
Material properties of the FGM constituents [simsek].

Properties	unit	Aluminum	Alumina(Al_2o_3)
E	*Gpa*	70	380
ρ	Kg/m^3	2702	3960
v	-	0.3	0.3

Table2
Comparison of the non-dimensional fundamental frequency for a C-F FG beam with various gradient indexes.

Power-law Exponent	L/h	Present	Simsek[38](2010)	Pradhan&Chakraverty[39](2014)
		Analytical	Lagrange's equations	Rayleigh-Ritz
p=0	5	1.88995	1.89479	1.9021
	20	1.93367	1.94957	1.9501
p =0.2	5	1.75994	1.76637	1.7717
	20	1.80425	1.81456	1.8147
p =0.5	5	1.60374	1.61817	1.6233
	20	1.65367	1.66044	1.6612
p =1	5	1.44874	1.46300	1.4688
	20	1.48197	1.50104	1.5025
p =2	5	1.30147	1.33376	1.3396
	20	1.31789	1.36968	1.3715

In Tables 3, 4 and 5., a comparison between natural frequency of the clamped-free FG Nano beams with carried concentrated mass are presented for various values of the gradient index (*p*=0,0.2,0.5,1,2,), nonlocal parameters ($\mu = 0,1,2,3$) , different mass ($m_{tipmass} = (0,2,4,8)*10^{-6} kg$ and three different values of aspect ratio ($L/h = 40,50,60$) based on analytical solution method. Clamped-free boundary condition predicates the edge conditions at $x = 0$, $x = L$ of the beam. As seen in tables, by fixing the nonlocal parameter and varying the material distribution parameter, the fundamental frequencies decrease. By increasing the power law exponents from zero changes the composition of the FG beam from a fully ceramic beam to a beam with a combination of ceramic and metal. In other words, the percentage of metal phase increases by increasing power index. So this Occurrence leads to the increment in flexibility of the FG beams by smaller value of Young's modulus. Thus, as also known from mechanical vibrations, natural frequencies decrease as the flexibility of a structure increases. However, the increasing of nonlocal parameter causes the decreasing in fundamental frequency, at a constant material graduation index. The first dimensionless natural frequency of the nano beam with the tip mass and clamped-free boundary condition has been presented at Table 3.

Table 3
Material graduation and nonlocality parameter effects on the natural frequency of a C-F FG Nano beams carried concentrated mass ($L/h = 20$).

μ	$m_{tipmass}$	Gradient index				
		0	0.2	0.5	1	2
0	0	1.93367	1.80425	1.70567	1.60097	1.49989
	2	1.87476	1.77007	1.54012	1.55458	1.30876
	4	1.71421	1.692014	1.50587	1.40134	1.29998
	8	1.647954	1.52154	1.47895	1.38745	1.18785
1	0	1.4224	1.38745	1.24784	1.14789	0.98745
	2	1.35547	1.24789	1.12475	0.97854	0.87985
	4	1.21458	1.19145	1.00094	0.92134	0.78794
	8	1.12574	1.02145	0.98745	0.88954	0.67489
2	0	1.00478	0.97854	0.78945	0.67498	0.60789
	2	0.98745	0.9320	0.71245	0.58746	0.50756
	4	0.88746	0.80654	0.61465	0.49899	0.46987
	8	0.75847	0.70946	0.59587	0.428963	0.37896
3	0	0.78921	0.687455	0.57841	0.456978	0.40741
	2	0.647895	0.547984	0.44546	0.400124	0.39640
	4	0.54711	0.456987	0.38748	0.354120	0.30078
	8	0.45814	0.404568	0.35467	0.301460	0.29870

By studying the results of Table 3, it is observed that fundamental frequency will be decreased by increasing nonlocal parameters for every gradient index. This decrease in frequency value emphasizes on the importance of size effect. Also it is obvious from this table that increasing mass density yields decreasing of natural frequencies for every types of gradient indexes; thus mass density has a significant effect on the dimensionless natural frequencies. As we know, increasing of the power indexes lead to rise the percentage of metal phase and thereupon FG beams will be more flexible and fundamental frequency values reduce.

Also, Table 4. contains the effect of nonlocal parameters, gradient indexes and mass density on natural frequencies of the FG Nano beams of clamped-free boundary condition with different gradient indexes for (L / h =50). It is obvious that increasing gradient index and nonlocal parameters yields to decrease natural frequency values.

Table 4
Material graduation and nonlocality parameter effects on the natural frequency of a C-F FG Nano beam with different tip mass ($L / h = 50$).

μ	$m_{tipmass}$	Gradient index				
		0	0.2	0.5	1	2
0	0	2.13367	2.10025	1.82464	1.69196	1.51149
	2	2.01478	1.97485	1.87456	1.64785	1.41478
	4	1.94578	1.87457	1.79452	1.621457	1.37891
	8	1.87459	1.74589	1.684753	1.502147	1.33147
1	0	1.78546	1.76541	1.56310	1.34789	1.28745
	2	1.68457	1.65410	1.53846	1.308954	1.23541
	4	1.57489	1.48793	1.45687	1.24587	1.14789
	8	1.45876	1.38749	1.21458	1.14587	1.09874
2	0	1.35478	1.27895	1.12458	1.021458	0.98745
	2	1.21487	1.20005	1.02478	0.98740	0.95014
	4	1.12547	1.09789	0.98745	0.87459	0.71950
	8	1.02145	0.94780	0.87456	0.78542	0.67965
3	0	1.00745	0.91478	0.87459	0.74872	0.72987
	2	1.03746	0.9.0745	0.84598	0.65459	0.50458
	4	0.93954	0.87460	0.73458	0.64125	0.40456
	8	0.84781	0.73412	0.64127	0.59874	0..36874

At last, the fundamental frequency is presented at Table 5. for FG Nano beam carried concentrated mass with different power-low indexes, clamped-free boundary conditions for $(L / h = 60)$. The conclusions that derived from this table for the effect of the tip masses and power index parameters on the natural frequency are similar to two previously tables.

Table 5
Material graduation and nonlocality parameter effects on the natural frequency of a C-F FG Nano beam with different tip mass ($L / h = 60$).

μ	$m_{tipmass}$	Gradient index				
		0	0.2	0.5	1	2
0	0	2.33367	2.20025	2.12464	1.89196	1.61149
	2	2.21458	2.145879	2.01236	1.78546	1.45871
	4	2.12458	2.04587	1.90745	1.65478	1.35484
	8	1.98745	1.81247	1.74589	1.47589	1.14785
1	0	1.98745	1.90247	1.80457	1.45876	1.21245
	2	1.84579	1.77459	1.70489	1.32145	1.12459
	4	1.65478	1.56984	1.45478	1.23456	0.99878
	8	1.45879	1.30216	1.24698	1.12365	0.89745
2	0	1.58745	1.40247	1.30457	1.15876	0.98745
	2	1.45469	1.321465	1.224587	1.14789	1.01456
	4	1.325478	1.254134	1.12456	1.09874	0.99544
	8	1.012354	0.99874	0.87456	0.78450	0.67456
3	0	1.18745	1.00247	0.97456	0.7891	0.64579
	2	1.05014	0.98741	0.87452	0.69512	0.54123
	4	0.98741	0.84452	0.78541	0.54169	0.45690
	8	0.874165	0.78214	0.65478	0.45621	0.314569

Therefore by comparing the frequency values for FG Nano beams with tip mass for a prescribed material properties, mass density and gradient indexes in Tables 3-5., can observe the influence of slenderness on frequencies. It is concluded that by increasing of L / h , the natural frequencies will be increased.

5 CONCLUSIONS

In this study, the exact form equations for the analysis of the transverse vibration modes of a FG Timoshenko nano beam carrying concentrated mass are obtained. After imposing Hamilton's principle on the Timoshenko's beam element, equations of motion are attained. Then time variable function is separated from equations of motion. Later, two coupled equations of motion are combined, a homogeneous differential equation is extracted with respect to transverse displacement, and undefined coefficients method is used to solve differential equation. A numerical study of a concentrated mass on Timoshenko nano beam was conducted to investigate mass intensity on the nano beam natural frequencies.

According to the numerical results, it is revealed that the effect of different parameters are investigated, the effect of gradient indexes, nonlocal parameters, rotatory inertia, material property gradient index and mass density on fundamental frequencies of nano FG beams are investigated. It is concluded that various factors such as nonlocal parameter, gradient index, mass density, material properties and aspect ratio play important roles in dynamic behavior of FG Nano beams.

- It is concluded that increasing in gradient indexes is cause of decreasing of fundamental frequency.
- Also it is revealed that increasing of slenderness aspect ratio yields the increase in fundamental frequencies for every value of power indexes.
- It is illustrated that presence of nonlocality leads to reduction in natural frequency.
- It is concluded that the mass attached at free end of the beam yields reduction in natural frequencies and by increasing in the mass density of the masse the dimensionless frequency will be lower.

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